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Signed Product Cordial of the Sum and Union of Two Fourth Power of Paths and Cycles

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Abstract: A simple graph is said to be signed product cordial if it admits ±1 labeling that satisfies certain conditions. Our aim in this paper is to contribute some new results on signed product cordial labeling and present necessary and sufficient conditions for signed product cordial of the sum and union of two fourth power of paths. We also study the signed product cordiality of the sum and union of fourth power cycles The residue classes modulo 4 are accustomed to find suitable labelings for each class to achieve our task. We have shown that the union and the join of any two fourth power of paths are always signed product cordial. Howover, the join and union of fourth power of cycles are only signed codial with some expectional situations.

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1. Introduction

It is known that the graph theory and its branches have become interest topics for almost all fields of mathematics and also other area of science such as Chemistry, Biology, Physics, Communication, Economics, Engineering, Operator research and especially computer science [1, 3]. We begin with finite, connected and undirected graph $G = (V, E)$ without loops and multiple edges. Here elements of sets $V(G)$ and $E(G)$ are known as vertices and edges respectively. The concept of graph labeling was introduced by Rosa in 1967 [9]. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain condition (s). Most of the graph labeling problems have the following three common characteristics: a set of numbers for assignment of vertex labels, a rule that assigns a label to each edge and some condition (s) that these labels must satisfy. For detailed survey on graph labeling we refer to A Dynamic Survey of Graph Labeling by Gallian [8]. Cordial labeling was introduced by Cahit [2] who called a graph G cordial if there is a vertex labeling $f: V(G) \rightarrow \{0, 1\}$ such that the induced labeling $f^*: E(G) \rightarrow \{0, 1\}$, defined by $f^*(xy) = |f(x) - f(y)|$, for

all edges $x, y \in E(G)$ and with the following inequalities holding: $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$, where $v_f(i)$ (respectively $e_f(i)$) is the number of vertices (respectively, edges) labeled with i. Motivated through the concept of cordial labeling the product cordial labeling was introduced by Sundaram et al [11] where absolute difference of vertex labels is replaced by product of vertex labels. Some researchers have written many papers in this field [10, 12].

In our work we test the signed product cordial of the sum of two fourth power of paths and cycles $P_n^4 + P_m^4$ and $C_n^4 + C_m^4$ and also the union of two fourth power of paths and cycles $P_n^4 \cup P_m^4$ and $C_n^4 \cup C_m^4$.

2. Preliminaries

Definition 1. A vertex labeling of graph G $f: V(G) \rightarrow \{-1, 1\}$ with induced edge labeling $f^*: E(G) \rightarrow \{-1, 1\}$ defined by $f^*(uv) = f(u).f(v)$ is called a signed product cordial labeling if $|v_f(-1) - v_f(1)| \leq 1$ and $|e_f(-1) - e_f(1)| \leq 1$, where $v_f(-1)$ is the number of vertices labeled with -1, $v_f(1)$ is the number of vertices labeled 1, $e(f^*)(-1)$ is the number of edges labeled with -1 and

$e(f^*)(1)$ is the number of edges labeled 1. A graph G is signed product cordial if it admits signed product cordial labeling.

Definition 2. The Sum or join is $G_1 + G_2$, is the graph with vertex set and edge set given by $V(G_1) + V(G_2)$ $= V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup J$, where J consists of edges join each vertex of G_1 to every vertex of G_2.

Definition 3. The union is $G_1 \cup G_2 = (V(G_1) \cup V(G_2), E_1 \cup E_2)$ (simple graph).

Definition 4. The fourth power of a cycles C_n denoted by C_n^4 is $C_n \cup J$, where J is the set of all edges of the form edges $v_i v_j$ such that $2 \le d(v_i v_j) \le 4$ and $i < j$.

Diab [2-3] has reported several results concerning the sum and union of the cycles C_n and paths P_m together and with other specific graphs.

3. Terminologies and Notations

A cycle with n vertices and n edges is denoted by C_n, and its fourth power C_n^4 has n vertices and $4n - 9$ edges. We let L_{4r} denote the labeling $(-1)_2 11$ $(-1)_2 11 \ldots (-1)_2 11$ (repeated r-times) and let L'_{4r} denote the labeling $(-1)11(-1)$ $(-1)11(-1) \ldots (-1)11(-1)$ (repeated r -times). The labeling $11(-1)_2$ $11(-1)_2 \ldots 11(-1)_2$ (repeated r -times) and $1(-1)_2 11(-1)_2 1 \ldots 1(-1)_2 1$ (repeated r-times) are written S_{4r} and S'_{4r} respectively. Let M_r denote the labeling $(-1)1$ $(-1)1 \ldots (-1)1$, zero-one repeated r times if r is even and $(-1)1$ $(-1)1 \ldots (-1)1(-1)$ if r is odd; for example, $M_6 = (-1)1(-1)1(-1)1$ and $M_5 = (-1)1(-1)1(-1)$.

We let M'_r denote the labeling $1(-1)1(-1) \ldots 1(-1)$. Sometimes, we modify the labeling M_r or M'_r by adding symbols at one end or the other (or both). Also, L_{4r} (or L'_{4r}) with extra labeling from right or left (or both sides). If L is a labeling for fourth power of paths P_m and M is a labeling for fourth power of paths P_n, then we use the notation $[L; M]$ to represent the labeling of the sum $P_m^4 + P_n^4$. Additional notation that we use is the following. For a given labeling of the sum $P_m^4 + P_n^4$, we let v_i and e_i (for $i = -1, 1$) be the numbers of labels that are i as before, we let x_i and a_i be the corresponding quantities for P_m^4, and we let y_i and b_i be those for P_n^4. It is easy to verify that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1)$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) + (x_{-1} - x_1)(y_{-1} - y_1)$.

Also for $P_m^4 \cup P_n^4$, we use the same notation $[L; M]$. Additional notation that we use is the following. For a given labeling of the union $P_m^4 \cup P_n^4$, we let v_i and e_i (for $i = -1, 1$) be the numbers of labels that are i as before, we let x_i and a_i be the corresponding quantities for P_m^4, and we let y_i and b_i be those for P_n^4. It is easy to verify that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1)$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1)$. When it comes to the proof, we only need to show that, for each specified combination of labeling, $|v_{-1} - v_1| \le 1$ and $|e_{-1} - e_1| \le 1$.

4. Main Results

4.1. Signed Product Cordial of the Sum of Two Fourth Power of Paths

In this subsection, we prove that the signed product cordial for the sum of two fourth power of paths. For this purpose let us study the following lemmas.

Lemma 4.1.1. If $n \equiv 0 (mod 4)$, then $P_n^4 + P_m^4$ is signed product cordial for all $m, n \ge 7$.

Proof. Suppose that $n = 4r$, where $r \ge 2$. The following cases will be examined.

Case 1. $m \equiv 0 (mod 4)$.

Suppose that $m = 4s$, where $s \ge 2$. Then we label the vertices of $P_{4r}^4 + P_{4s}^4$ by $[-1L_{4r-4} - 111; 1_2 L'_{4s-4} - 1_2]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = a_1 = 8r - 5, y_{-1} = y_1 = 2s, b_{-1} = b_1 = 8s - 5$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) + (x_{-1} - x_1)(y_{-1} - y_1) = 0$. As an example, Figure 1 illustrates $P_8^4 + P_8^4$. Hence, $P_{4r}^4 + P_{4s}^4$ is signed product cordial.
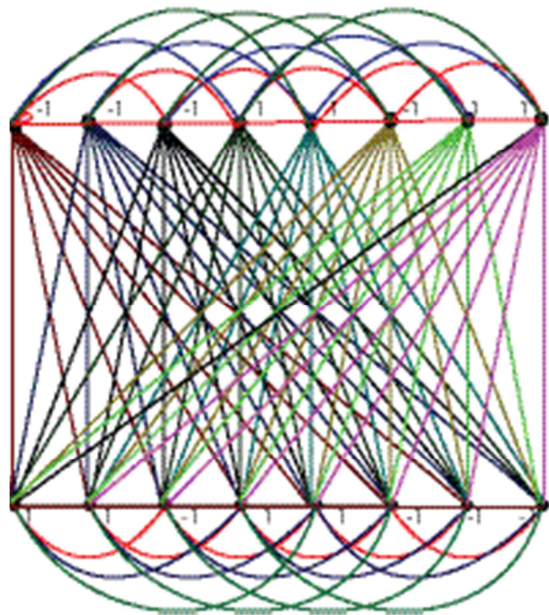

Figure 1. $P_8^4 + P_8^4$.

Case 2. $m \equiv 1 (mod 4)$.

Suppose that $m = 4s + 1$, where $s \ge 2$. Then we label the vertices of $P_{4r}^4 + P_{4s+1}^4$ by $[-1L_{4r-4} - 111; -1_2 L_{4s-4} 1 - 11]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = a_1 = 8r - 5, y_{-1} = 2s + 1, y_1 = 2s, b_{-1} = b_1 = 8s - 3$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r}^4 + P_{4s+1}^4$ is signed product cordial.

Case 3. $m \equiv 2 (mod 4)$.

Suppose that $m = 4s + 2$, where $s \ge 2$. Then we label the vertices of $P_{4r}^4 + P_{4s+2}^4$ by $[-1L_{4r-4} - 111; -11_3 - 1S_{4s-4} - 1]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = a_1 = 8r - 5, y_{-1} = y_1 = 2s + 1, b_{-1} = b_1 = 8s - 1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r}^4 + P_{4s+2}^4$ is signed product cordial.

Case 4. $m \equiv 3 (mod 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $P_{4r}^4 + P_{4s+3}^4$ by $[-1L_{4r-4} - 111; -1_2 1L_{4s}]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = a_1 = 8r - 5, y_{-1} = 2s + 2, y_1 = 2s + 1, b_{-1} = b_1 = 8s + 1$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r}^4 + P_{4s+3}^4$ is signed product cordial.

Lemma 4.1.2. If $n \equiv 1 (mod 4)$, then $P_n^4 + P_m^4$ is signed product cordial for all $m, n \geq 7$.

Proof. Suppose that $n = 4r + 1$, where $r \geq 2$. The following cases will be examined.

Case 1. $m \equiv 1 (mod 4)$.

Suppose that $m = 4s + 1$, where $s \geq 2$. Then we label the vertices of $P_{4r+1}^4 + P_{4s+1}^4$ by $[-1_2 L_{4r-4} 1 - 11; 1_2 L'_{4s-4} - 11 - 1]$. Therefore $x_{-1} = 2r + 1, x_1 = 2r, a_{-1} = a_1 = 8r - 3, y_{-1} = 2s, y_1 = 2s + 1, b_{-1} = b_1 = 8s - 3$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Hence, $P_{4r+1}^4 + P_{4s+1}^4$ is signed product cordial.

Case 2. $m \equiv 2 (mod 4)$.

Suppose that $m = 4s + 2$, where $s \geq 2$. Then we label the vertices of $P_{4r+1}^4 + P_{4s+2}^4$ by $[-1_2 L_{4r-4} 1 - 11; -11_3 - 1 S_{4s-4} - 1]$. Therefore $x_{-1} = 2r + 1, x_1 = 2r, a_{-1} = a_1 = 8r - 3, y_{-1} = y_1 = 2s + 1, b_{-1} = b_1 = 8s - 1$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r+1}^4 + P_{4s+2}^4$ is signed product cordial.

Case 3. $m \equiv 3 (mod 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $P_{4r+1}^4 + P_{4s+3}^4$ by $[-1_2 L_{4r-4} 1 - 11; 1_2 S_{4s} - 1]$. Therefore $x_{-1} = 2r + 1, x_1 = 2r, a_{-1} = a_1 = 8r - 3, y_{-1} = 2s + 1, y_1 = 2s + 2, b_{-1} = b_1 = 8s + 1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Hence, $P_{4r+1}^4 + P_{4s+3}^4$ is signed product cordial.

Lemma 4.1.3. If $n \equiv 2 (mod 4)$, then $P_n^4 + P_m^4$ is signed product cordial for all $m, n \geq 7$.

Proof. Suppose that $n = 4r + 2$, where $r \geq 2$. The following cases will be examined.

Case 1. $m \equiv 2 (mod 4)$.

Suppose that $m = 4s + 2$, where $s \geq 2$. Then we label the vertices of $P_{4r+2}^4 + P_{4s+2}^4$ by $[-11_3 - 1 S_{4r-4} - 1; -11_3 - 1 S_{4s-4} - 1]$. Therefore $x_{-1} = x_1 = 2r + 1, a_{-1} = a_1 = 8r - 1, y_{-1} = y_1 = 2s + 1, b_{-1} = b_1 = 8s - 1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r+2}^4 + P_{4s+2}^4$ is signed product cordial.

Case 2. $m \equiv 3 (mod 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $P_{4r+2}^4 + P_{4s+3}^4$ by $[-11_3 - 1 S_{4r-4} - 1; -1_2 1 L_{4s}]$. Therefore $x_{-1} = x_1 = 2r + 1, a_{-1} = a_1 = 8r - 1, y_{-1} = 2s + 2, y_1 = 2s + 1, b_{-1} = b_1 = 8s + 1$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r+2}^4 + P_{4s+3}^4$ is signed product cordial.

Lemma 4.1.4. If $n \equiv 3 (mod 4)$, then $P_n^4 + P_m^4$ is signed product cordial for all $m, n \geq 7$.

Proof. Suppose that $n = 4r + 3$, where $r \geq 2$. The following cases will be examined.

Case 1. $m \equiv 3 (mod 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $P_{4r+3}^4 + P_{4s+3}^4$ by $[-1_2 1 L_{4r}; 1_2 S_{4s} - 1]$. Therefore $x_{-1} = 2r + 2, x_1 = 2r + 1, a_{-1} = a_1 = 8r + 1, y_{-1} = 2s + 1, y_1 = 2s + 2, b_{-1} = b_1 = 8s + 1$. It follows

that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = -1$. Hence, $P_{4r+3}^4 + P_{4s+3}^4$ is signed product cordial.

As a consequence of all lemmas mentioned above we conclude that

Theorem 4.1 The sum of two fourth Power of Paths $P_n^4 + P_m^4$ is signed product cordial for all $m, n \geq 7$.

4.2. Signed Product Cordial of Sum of Two Fourth Power of Cycles

Here, we study The signed product cordial of sum of two fourth Power of cycles.

Lemma 4.2.1. If $n \equiv 0 (mod 4)$, then $C_n^4 + C_m^4$ is signed product cordial for all $m, n \geq 7$.

Proof. Suppose that $n = 4r$, where $r \geq 2$. The following cases will be examined.

Case 1. $m \equiv 0 (mod 4)$.

Suppose that $m = 4s$, where $s \geq 2$. Then we label the vertices of $C_{4r}^4 + C_{4s}^4$ by $[S'_{4r}; 1_3 M_{4s-6} - 1_3]$ Therefore $x_{-1} = x_1 = 2r, a_{-1} = 8r - 4, a_1 = 8r - 5, y_{-1} = y_1 = 2s, b_{-1} = 8s - 5, b_1 = 8s - 4$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) + (x_{-1} - x_1)(y_{-1} - y_1) = 0$. Hence, $C_{4r}^4 + C_{4s}^4$ is signed product cordial.

Case 2. $m \equiv 1 (mod 4)$.

Suppose that $m = 4s + 1$, where $s \geq 2$. Then we label the vertices of $C_{4r}^4 + C_{4s+1}^4$ by $[1_3 M_{4r-6} - 1_3; L_{4s} - 1]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = 8r - 5, a_1 = 8r - 4, y_{-1} = 2s + 1, y_1 = 2s, b_{-1} = 8s - 2, b_1 = 8s - 3$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $C_{4r}^4 + C_{4s+1}^4$ is signed product cordial.

Case 3. $m \equiv 2 (mod 4)$.

Suppose that $m = 4s + 2$, where $s \geq 2$. Then we label the vertices of $C_{4r}^4 + C_{4s+2}^4$ by $[S'_{4r}; -1_3 1 - 11_3 M_{4s-6}]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = 8r - 4, a_1 = 8r - 5, y_{-1} = y_1 = 2s + 1, b_{-1} = 8s - 1, b_1 = 8s$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $C_{4r}^4 + C_{4s+2}^4$ is signed product cordial.

Case 4. $m \equiv 3 (mod 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $C_{4r}^4 + C_{4s+3}^4$ by $[1_3 M_{4r-6} - 1_3; L'_{4s} - 11 - 1]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = 8r - 5, a_1 = 8r - 4, y_{-1} = 2s + 2, y_1 = 2s + 1, b_{-1} = 8s + 2, b_1 = 8s + 1$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $C_{4r}^4 + C_{4s+3}^4$ is signed product cordial.

Lemma 4.2.2. If $n \equiv 1 (mod 4)$, then $C_n^4 + C_m^4$ is signed product cordial for all $m, n \geq 7$.

Proof. Suppose that $n = 4r + 1$, where $r \geq 2$. The following cases will be examined.

Case 1. $m \equiv 1 (mod 4)$.

Suppose that $m = 4s + 1$, where $s \geq 2$. Then we label the vertices of $C_{4r+1}^4 + C_{4s+1}^4$ by $[L_{4r} - 1; 1_3 L'_{4s-4} - 1_2]$. Therefore $x_{-1} = 2r + 1, x_1 = 2r, a_{-1} = 8r - 2, a_1 = 8r - 3, y_{-1} = 2s, y_1 = 2s + 1, b_{-1} = 8s - 3, b_1 = 8s - 2$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Hence, $C_{4r+1}^4 + C_{4s+1}^4$ is signed product cordial.

Case 2. $m \equiv 2 (mod 4)$.

Suppose that $m = 4s + 2$, where $s \geq 2$. Then we label the vertices of $C_{4r+1}^4 + C_{4s+2}^4$ by $[L_{4r} - 1; -1_3 1 - 11_3 M_{4s-6}]$.

Therefore $x_{-1} = 2r + 1, x_1 = 2r, a_{-1} = 8r - 2, a_1 = 8r - 3, y_{-1} = y_1 = 2s + 1, b_{-1} = 8s - 1, b_1 = 8s$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $C_{4r+1}^4 + C_{4s+2}^4$ is signed product cordial.

Case 3. $m \equiv 3(mod 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $C_{4r+1}^4 + C_{4s+3}^4$ by $[1_3 L'_{4r-4} - 1_2; L'_{4s} - 11 - 1]$. Therefore $x_{-1} = 2r, x_1 = 2r + 1, a_{-1} = 8r - 3, a_1 = 8r - 2, y_{-1} = 2s + 2, y_1 = 2s + 1, b_{-1} = 8s + 2, b_1 = 8s + 1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Hence, $C_{4r+1}^4 + C_{4s+3}^4$ is signed product cordial.

Lemma 4.2.3. If $n \equiv 2(mod 4)$, then $C_n^4 + C_m^4$ is signed product cordial for all $m, n \geq 7$.

Proof. Suppose that $n = 4r + 2$, where $r \geq 2$. The following cases will be examined.

Case 1. $m \equiv 2(mod 4)$.

Suppose that $m = 4s + 2$, where $s \geq 2$. Then we label the vertices   of $C_{4r+2}^4 + C_{4s+2}^4$ by $[-1_3 1_3 L'_{4r-4}; -1_3 1 - 11_3 M_{4s-6}]$. Therefore $x_{-1} = x_1 = 2r + 1, a_{-1} = 8r, a_1 = 8r - 1, y_{-1} = y_1 = 2s + 1, b_{-1} = 8s - 1, b_1 = 8s$. It follows   that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $C_{4r+2}^4 + C_{4s+2}^4$ is signed product cordial.

Case 2. $m \equiv 3(mod 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices   of $C_{4r+2}^4 + C_{4s+3}^4$ by $[-1_3 1 - 11_3 M_{4s-6}; L'_{4s} - 11 - 1]$ Therefore $x_{-1} = x_1 = 2r + 1, a_{-1} = 8r - 1, a_1 = 8r, y_{-1} = 2s + 2, y_1 = 2s + 1, b_{-1} = 8s + 2, b_1 = 8s + 1$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $C_{4r+2}^4 + C_{4s+3}^4$ is signed product cordial.

As a consequence of all lemmas mentioned in subsection 4.2 we conclude that

Theorem 4.1. The sum of two fourth Power of cycles $C_n^4 + C_m^4$ is signed product cordial for all $m, n \geq 7$ except at $(m, n) = (7, 7)$

4.3 Signed product cordial of union of two fourth Power of Paths

Here, we prove that signed product cordial of the union of two fourth Power of Paths. To achieve this result, let us study the following lemmas.

Lemma 4.3.1. If $n \equiv 0(mod 4)$, then $P_n^4 \cup P_m^4$ is signed product cordial for all $m, n \geq 7$.

Proof. Suppose that $n = 4r$, where $r \geq 2$. The following cases will be examined.

Case 1. $m \equiv 0(mod 4)$.

Suppose that $m = 4s$, where $s \geq 2$. Then we label the vertices of $P_{4r}^4 \cup P_{4s}^4$ by $[-1 L_{4r-4} - 111; 1_2 L'_{4s-4} - 1_2,]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = a_1 = 8r - 5, y_{-1} = y_1 = 2s, b_{-1} = b_1 = 8s - 5$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. As an example, Figure 2 illustrates $P_8^4 \cup P_8^4$. Hence, $P_{4r}^4 \cup P_{4s}^4$ is signed product cordial.

Case 2. $m \equiv 1(mod 4)$.

Suppose that $m = 4s + 1$, where $s \geq 2$. Then we label the vertices of $P_{4r}^4 \cup P_{4s+1}^4$ by $[-1 L_{4r-4} - 111; -1_2 L_{4s-4} 1 - 11]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = a_1 = 8r - 5, y_{-1} = 2s + 1, y_1 = 2s, b_{-1} = b_1 = 8s - 3$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r}^4 \cup P_{4s+1}^4$ is signed product cordial.
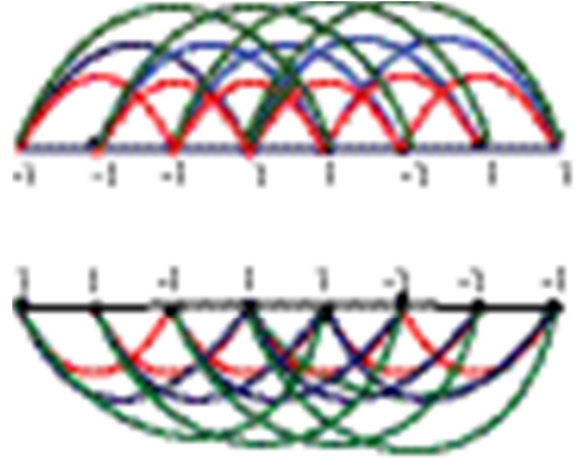

Figure 2. $P_8^4 \cup P_8^4$.

Case 3. $m \equiv 2(mod 4)$.

Suppose that $m = 4s + 2$, where $s \geq 2$. Then we label the vertices   of $P_{4r}^4 \cup P_{4s+2}^4$ by $[-1 L_{4r-4} - 111; -11_3 - 1 S_{4s-4} - 1]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = a_1 = 8r - 5, y_{-1} = y_1 = 2s + 1, b_{-1} = b_1 = 8s - 1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r}^4 \cup P_{4s+2}^4$ is signed product cordial.

Case 4. $m \equiv 3(mod 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices   of $P_{4r}^4 \cup P_{4s+3}^4$ by $[-1 L_{4r-4} - 111; -1_2 1 L_{4s}]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = a_1 = 8r - 5, y_{-1} = 2s + 2, y_1 = 2s + 1, b_{-1} = b_1 = 8s + 1$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r}^4 \cup P_{4s+3}^4$ is signed product cordial.

Lemma 4.3.2. If $n \equiv 1(mod 4)$, then $P_n^4 \cup P_m^4$ is signed product cordial for all $m, n \geq 7$.

Proof. Suppose that $n = 4r + 1$, where $r \geq 2$. The following cases will be examined.

Case 1. $m \equiv 1(mod 4)$.

Suppose that $m = 4s + 1$, where $s \geq 2$. Then we label the vertices of $P_{4r+1}^4 \cup P_{4s+1}^4$ by $[-1_2 L_{4r-4} 1 - 11; 1_2 L'_{4s-4} - 11 - 1]$. Therefore $x_{-1} = 2r + 1, x_1 = 2r, a_{-1} = a_1 = 8r - 3, y_{-1} = 2s, y_1 = 2s + 1, b_{-1} = b_1 = 8s - 3$. It follows   that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r+1}^4 \cup P_{4s+1}^4$ is signed product cordial.

Case 2. $m \equiv 2(mod 4)$.

Suppose that $m = 4s + 2$, where $s \geq 2$. Then we label the vertices   of $P_{4r+1}^4 \cup P_{4s+2}^4$ by $[-1_2 L_{4r-4} 1 - 11; -11_3 - 1 S_{4s-4} - 1]$. Therefore $x_{-1} = 2r + 1, x_1 = 2r, a_{-1} = a_1 = 8r - 3, y_{-1} = y_1 = 2s + 1, b_{-1} = b_1 = 8s - 1$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r+1}^4 \cup P_{4s+2}^4$ is signed product cordial.

Case 3. $m \equiv 3(mod 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $P_{4r+1}^4 \cup P_{4s+3}^4$ by $[-1_2 L_{4r-4} 1 - 11; 1_2 S_{4s} - 1]$. Therefore $x_{-1} = 2r + 1, x_1 = 2r, a_{-1} = a_1 = 8r - 3, y_{-1} = 2s + 1, y_1 = 2s + 2, b_{-1} = b_1 = 8s + 1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r+1}^4 \cup P_{4s+3}^4$ is signed product cordial.

Lemma 4.3.3. If $n \equiv 2(mod 4)$, then $P_n^4 \cup P_m^4$ is signed product cordial for all $m, n \geq 7$.

Proof. Suppose that $n = 4r + 2$, where $r \geq 2$. The following cases will be examined.

Case 1. $m \equiv 2(mod 4)$.

Suppose that $m = 4s + 2$, where $s \geq 2$. Then we label the vertices of $P_{4r+2}^4 \cup P_{4s+2}^4$ by $[-11_3 - 1S_{4r-4} - 1; -11_3 - 1S_{4s-4} - 1]$. Therefore $x_{-1} = x_1 = 2r + 1, a_{-1} = a_1 = 8r - 1, y_{-1} = y_1 = 2s + 1, b_{-1} = b_1 = 8s - 1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r+2}^4 \cup P_{4s+2}^4$ is signed product cordial.

Case 2. $m \equiv 3(mod 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $P_{4r+2}^4 \cup P_{4s+3}^4$ by $[-11_3 - 1S_{4r-4} - 1; -1_2 1L_{4s}]$. Therefore $x_{-1} = x_1 = 2r + 1, a_{-1} = a_1 = 8r - 1, y_{-1} = 2s + 2, y_1 = 2s + 1, b_{-1} = b_1 = 8s + 1$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r+2}^4 \cup P_{4s+3}^4$ is signed product cordial.

Lemma 4.3.4. If $n \equiv 3(mod 4)$, then $P_n^4 \cup P_m^4$ is signed product cordial for all $m, n \geq 7$.

Proof. Suppose that $n = 4r + 3$, where $r \geq 2$. The following cases will be examined.

Case 1. $m \equiv 3(mod 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $P_{4r+3}^4 \cup P_{4s+3}^4$ by $[-1_2 1L_{4r}; 1_2 S_{4s} - 1]$. Therefore $x_{-1} = 2r + 2, x_1 = 2r + 1, a_{-1} = a_1 = 8r + 1, y_{-1} = 2s + 1, y_1 = 2s + 2, b_{-1} = b_1 = 8s + 1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r+3}^4 \cup P_{4s+3}^4$ is signed product cordial.

As a consequence of all lemmas proved in subsection 4.3 we conclude that

Theorem 4.3 The union of two fourth Power of Paths $P_n^4 \cup P_m^4$ is signed product cordial for all $m, n \geq 7$.

4.4 Signed product cordial of union of two fourth Power of cycles

Here, we prove that the signed product cordial of the union of two fourth Power of cycles. For this purpose let us study the following lemmas.

Lemma 4.4.1. If $n \equiv 0(mod 4)$, then $C_n^4 \cup C_m^4$ is signed product cordial for all $m, n \geq 7$.

Proof. Suppose that $n = 4r$, where $r \geq 2$. The following cases will be examined.

Case 1. $m \equiv 0(mod 4)$.

Suppose that $m = 4s$, where $s \geq 2$. Then we label the vertices of $C_{4r}^4 \cup C_{4s}^4$ by $[S'_{4r}; 1_3 M_{4s-6} - 1_3]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = 8r - 4, a_1 = 8r - 5, y_{-1} = y_1 = 2s, b_{-1} = 8s - 5, b_1 = 8s - 4$. It follows that $v_{-1} - v_1 = (x_{-1} - x_1) + (y_{-1} - y_1) = 0$ and $e_{-1} - e_1 = (a_{-1} - a_1) + (b_{-1} - b_1) = 0$. Hence, $C_{4r}^4 \cup C_{4s}^4$ is signed product cordial.

Case 2. $m \equiv 1(mod 4)$.

Suppose that $m = 4s + 1$, where $s \geq 2$. Then we label the vertices of $C_{4r}^4 \cup C_{4s+1}^4$ by $[1_3 M_{4r-6} - 1_3; L_{4s} - 1]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = 8r - 5, a_1 = 8r - 4, y_{-1} = 2s + 1, y_1 = 2s, b_{-1} = 8s - 2, b_1 = 8s - 3$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $C_{4r}^4 \cup C_{4s+1}^4$ is signed product cordial.

Case 3. $m \equiv 2(mod 4)$.

Suppose that $m = 4s + 2$, where $s \geq 2$. Then we label the vertices of $C_{4r}^4 \cup C_{4s+2}^4$ by $[S'_{4r;3} 1 - 11_3 M_{4s-6}]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = 8r - 4, a_1 = 8r - 5, y_{-1} = y_1 = $

$2s + 1, b_{-1} = 8s - 1, b_1 = 8s$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $C_{4r}^4 \cup C_{4s+2}^4$ is signed product cordial.

Case 4. $m \equiv 3(mod 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $C_{4r}^4 \cup C_{4s+3}^4$ by $[1_3 M_{4r-6} - 1_3; L'_{4s} - 11 - 1]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = 8r - 5, a_1 = 8r - 4, y_{-1} = 2s + 2, y_1 = 2s + 1, b_{-1} = 8s + 2, b_1 = 8s + 1$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $C_{4r}^4 \cup C_{4s+3}^4$ is signed product cordial.

Lemma 4.4.2. If $n \equiv 1(mod 4)$, then $C_n^4 \cup C_m^4$ is signed product cordial for all $m, n \geq 7$.

Proof. Suppose that $n = 4r + 1$, where $r \geq 2$. The following cases will be examined.

Case 1. $m \equiv 1(mod 4)$.

Suppose that $m = 4s + 1$, where $s \geq 2$. Then we label the vertices of $C_{4r+1}^4 \cup C_{4s+1}^4$ by $[L_{4r} - 1; 1_3 L'_{4s-4} - 1_2]$. Therefore $x_{-1} = 2r + 1, x_1 = 2r, a_{-1} = 8r - 2, a_1 = 8r - 3, y_{-1} = 2s, y_1 = 2s + 1, b_{-1} = 8s - 3, b_1 = 8s - 2$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $C_{4r+1}^4 \cup C_{4s+1}^4$ is signed product cordial.

Case 2. $m \equiv 2(mod 4)$.

Suppose that $m = 4s + 2$, where $s \geq 2$. Then we label the vertices of $C_{4r+1}^4 \cup C_{4s+2}^4$ by $[L_{4r} - 1; -1_3 1 - 11_3 M_{4s-6}]$. Therefore $x_{-1} = 2r + 1, x_1 = 2r, a_{-1} = 8r - 2, a_1 = 8r - 3, y_{-1} = y_1 = 2s + 1, b_{-1} = 8s - 1, b_1 = 8s$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $C_{4r+1}^4 \cup C_{4s+2}^4$ is signed product cordial.

Case 3. $m \equiv 3(mod 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $C_{4r+1}^4 \cup C_{4s+3}^4$ by $[1_3 L'_{4r-4} - 1_2; L'_{4s} - 11 - 1]$. Therefore $x_{-1} = 2r, x_1 = 2r + 1, a_{-1} = 8r - 3, a_1 = 8r - 2, y_{-1} = 2s + 2, y_1 = 2s + 1, b_{-1} = 8s + 2, b_1 = 8s + 1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $C_{4r+1}^4 \cup C_{4s+3}^4$ is signed product cordial.

Lemma 4.4.3. If $n \equiv 2(mod 4)$, then $C_n^4 \cup C_m^4$ is signed product cordial for all $m, n \geq 7$.

Proof. Suppose that $n = 4r + 2$, where $r \geq 2$. The following cases will be examined.

Case 1. $m \equiv 2(mod 4)$.

Suppose that $m = 4s + 2$, where $s \geq 2$. Then we label the vertices of $C_{4r+2}^4 \cup C_{4s+2}^4$ by $[-1_3 1_3 L'_{4r-4}; -1_3 1 - 11_3 M_{4s-6}]$. Therefore $x_{-1} = x_1 = 2r + 1, a_{-1} = 8r, a_1 = 8r - 1, y_{-1} = y_1 = 2s + 1, b_{-1} = 8s - 1, b_1 = 8s$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $C_{4r+2}^4 \cup C_{4s+2}^4$ is signed product cordial.

Case 2. $m \equiv 3(mod 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $C_{4r+2}^4 \cup C_{4s+3}^4$ by $[-1_3 1 - 11_3 M_{4s-6}; L'_{4s} - 11 - 1]$. Therefore $x_{-1} = x_1 = 2r + 1, a_{-1} = 8r - 1, a_1 = 8r, y_{-1} = 2s + 2, y_1 = 2s + 1, b_{-1} = 8s + 2, b_1 = 8s + 1$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 0$. Hence, $C_{4r+2}^4 \cup C_{4s+3}^4$ is signed product cordial.

As a consequence of all lemmas proved in subsection 4.4 we conclude that.

Theorem 4.4 The union of two fourth Power of Paths $C_n^4 \cup C_m^4$ is signed product cordial for all $m, n \geq 7$ except at $(n, m) = (7, 7)$.

5. Conclusion

In this paper we test the signed product cordial of the sum and union of two fourth power of paths and also for cycles, we found that $P_n^4 + P_m^4$ and $P_n^4 \cup P_m^4$ is signed product cordial for all m, n, and also $C_n^4 + C_m^4$ and $C_n^4 \cup C_m^4$ is signed product cordial for all m, n except at $(n, m) = (7, 7)$.

In our next work we will continue studying the signed product cordial of the sum and union of high powers of paths, cycles and other graphs.

Acknowledgements

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