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Review Article

Vertex Colorings of Graph and Some of Their Applications in Promoting Global Competitiveness for National Growth and Productivity

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Abstract: This paper studies various results on vertex colorings of simple connected graphs, chromatic number, chromatic polynomials and some Algebraic properties of chromatic polynomials. Results were obtained on the roots of chromatic polynomials of simple connected graphs based on Read's conjecture. The chromatic number of every graph is the minimum number of colors to properly color the graph. Chromatic polynomial of a graph is a polynomial in integer and the leading coefficient of chromatic polynomial of a graph of order *n* and size *m* is always 1, whose coefficient alternate in sign. Through the application of famous graph theorem (the hand shaking lemma) by whiskey which states that: "the order of a graph twice its size". Hence, every graph has a chromatic polynomial but not all polynomials are chromatic. For example, the polynomial $\lambda^5 - 11\lambda^4 + 14\lambda^3 - 6\lambda^2 + 2\lambda$ is a polynomial for a graph on five vertices and eleven edges which does not exists. Because the maximum number size for a graph of order five is ten. The paper equally gave some practical applications of Vertex coloring in real life situations such as scheduling, allocation of channels to television and radio stations, separation of chemicals and traffic light signals.

Keywords: Adjacent Vertices, Chromatic Number, Chromatic Polynomials

1. Introduction

A graph G is a finite nonempty set V of objects called vertices (the singular is vertex) together with a set E of 2-element subsets of V called edges. Vertices are sometimes called points or nodes, while edges are sometimes referred to as lines or links. Each edge $\{u, v\}$ of V is commonly denoted by uv or vu. If $e = uv$, then the edge e is said to join u and v. The number of vertices in a graph G is the order of G and the number of edges is the size of G. We often use n for the order of a graph and m for its size. To indicate that a graph G has vertex set V and edge set E, we sometimes write $G = (V, E)$. To emphasize that V is the vertex set of a graph G, we often write V as $V(G)$ or V_G. For the same reason, we also write E as $E(G)$ [1].

A graph of order 1 is called a trivial graph and so a nontrivial graph has two or more vertices. A graph of size 0 is an empty graph and so a nonempty graph has one or more edges. Graphs are typically represented by diagrams in which each vertex is represented by a point or small circle (open or solid) and each edge is represented by a line segment or curve joining the corresponding small circles. A diagram that represents a graph G is referred to as the graph G itself and the small circles and lines representing the vertices and edges of G are themselves referred to as the vertices and edges of G. The first results about graph coloring deals almost exclusively with planar graphs in the form of the coloring of maps. While trying to color a map of the counties of England, Francis Guthrie postulated the four color conjecture, noting that four colors were sufficient to color the map so that no regions sharing a common border received the same color [9].

Guthrie's brother passed on the question to his mathematics teacher Augustus de Morgan at University College, who mentioned it in a letter to William Hamilton. Then Arthur Clayey raised the problem at a meeting of the London Mathematical Society. The same year, Alfred Kempe published a paper that claimed to establish the result, and for a decade the four color problem was considered solved. For his accomplishment Kempe was elected a Fellow of the Royal Society and later President of the London Mathematical Society. Heawood pointed out that Kempe's argument was wrong. Heawood himself modified that thought [10].

2. Vertex Colorings

The vertices of a graph G can also be classified using colorings. These colorings tell that certain vertices have a common property (or that they are similar in some aspect), if they share the same color. In this paper, we shall concentrate on proper vertex colorings, where adjacent vertices get different colors. In graph labeling usually integer number is given to an edge, or vertex, or to both i.e. to an edge and to a vertex of a graph. Similarly, in graph theory, we use some colors to label the edges or vertices. But there are some restrictions on using colors. The problem is, if we have n colors, then we have to find a way for coloring vertices such that no two adjacent vertices have the same color. There exists some other graph coloring problems also, for example, Edge Coloring and Face coloring. In edge coloring, not a single vertex is connected to two edges which are having same color. And face coloring is related to Geographical map coloring. Edge coloring and face coloring problems can be transmitted to vertex coloring [4].

2.1. Vertex Coloring

A function a: $V_G \rightarrow K$ is a vertex coloring of G by a set K of colors. A graph is said to be k vertex colorable or (k-colorable) if is possible to assign one color from a set of k colors to each vertex such that no two adjacent vertices have the same color. If the graph G is k colorable, but not ($k-1$) colorable we say that G is k chromatic graph.

2.2. Color Class

A color class in vertex coloring of a graph G is a subset of V_G containing all the vertices of a given color.

2.3. Proper Vertex Coloring

A proper vertex coloring of a graph G is a vertex coloring such that the end points of each edge are assigned two different colors [8].

Remark 1:

Quite commonly, it is implicit from context that the color creations are proper in which case each color class is an independent set of vertices. Example is the vertex coloring shown below which demonstrates that the graph G is 4-colorable [7].

$G =$

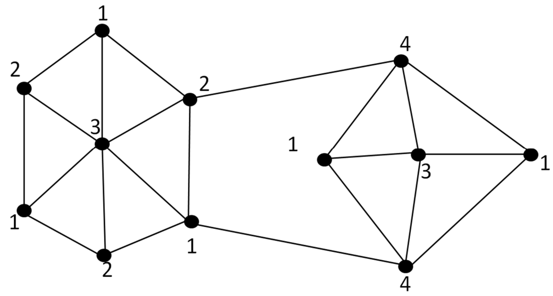

Figure 1. *Proper vertex 4-coloring of a graph G.*

2.4. (Vertex) Chromatic Number

The vertex chromatic number of a graph G, denoted as χ (G) is the minimum number of different colors require for a proper vertex-coloring of G. a graph G is (Vertex) k-chromatic if χ (G)=k.

The 3-coloring below shows that the graph G is 3-colorable, which means that χ (G) $\leq$ 3.

However the graph G contains three mutually adjacent vertices and hence is not 2-colorable. Thus G is three chromatic [5].

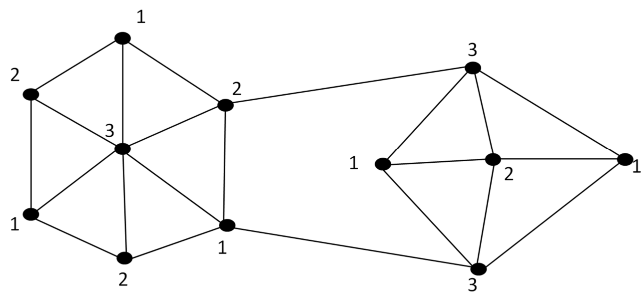

Figure 2. *A proper 3-coloring of the graph G.*

Proposition 1: let H be a subgraph of G. then χ (G) $\geq \chi$ (H)

Proof: whatever colors are used on the vertices of subgraph H in a minimum coloring of graph G can also be used in a coloring of H by itself.

The Join G+H of the graph G and H are obtained from the graph union $G \cup H$ by adding an edge between each vertex of G and each vertex of H.

Proposition 2: the join of G and H has chromatic number χ ($G + H$) = χ (G) + χ (H).

Proof: lower bound. In the join G+ H, no color used on the graph G can be the same as a color used on the subgraph H, since every vertex of G is adjacent to every vertex of H. since χ (G) colors are required for subgraph G and χ (H) colors are required for subgraph H, it follows that χ ($G + H$) $\geq \chi$ (G) + χ (H).

Upper Bound: Just use any χ (G) colors to properly color the subgraph G of $G + H$, and use χ (H) different colors to color the subgraph H.

2.5. Chromatic Polynomials (Chronomials)

During the period that the Four Color Problem was unsolved, which spanned more than a century, many approaches were introduced with the hopes that they would lead to a solution of this famous problem. Birkhoff (1912) defined a function $P\ (M, \lambda\ )$ that gives the number of proper λ-colorings of a map M for a positive integer λ. As we will see, $P\ (M, \lambda)$ is a polynomial in λ for every map M and is called the chromatic polynomial of M. Consequently, if it could be verified that $P\ (M, 4) > 0$ for every map M, then this would have established the truth of the Four Color Conjecture. Whitney (1932) expanded the study of chromatic polynomials from maps to graphs. While Whitney obtained a number of results on chromatic polynomials of graphs and others obtained results on the roots of chromatic polynomials of planar graphs, this did not contribute to a proof of the Four Color Conjecture. Renewed interest in chromatic polynomials of graphs occurred. Read (1968) wrote a survey paper on chromatic polynomials. For a graph G and a positive integer λ, the number of different proper λ-colorings of G is denoted by $P\ (G, \lambda)$ and is called the chromatic polynomial of G. Two λ-colorings c and c' of G from the same set $\{1, 2,..., \lambda\}$ of λ colors are considered different if $c(v) \neq c'(v)$ for some vertex v of G. Obviously, if $\lambda < \lambda\ (G)$, then $P\ (G,\ \lambda\ ) = 0$ [5].

There are some classes of graphs G for which $P\ (G, \lambda)$ can be easily computed.

Theorem 1. for every positive integer λ,

(a) $P\ (K_n, \lambda\ ) = \lambda\ (\lambda - 1)(\lambda - 2) \cdots (\lambda - n + 1) = \lambda^{(n)}$,

(b) $P\ (\overline{K}_n, \lambda) = \lambda^{n}$.

In particular, if $\lambda \geq n$ in for every positive integer,

$$P\ (Kn,\ ) = (-1)(-2) \cdots (-n + 1) = (n),\ \text{then}\ P\ (K_n, \lambda) = \lambda^{(n)} = \lambda!/(\lambda\text{-}n)!$$

We now determine the chromatic polynomial of C_4 in Figure 3, There are λ choices for the color of v_1. The vertices v_2 and v_4 must be assigned colors different from that assigned to v_1. The vertices v_2 and v_4 may be assigned the same color or may be assigned different colors. If v_2 and v_4 are assigned the same color, then there are $\lambda - 1$ choices for that color. The vertex v_3 can then be assigned any color except the color assigned to v_2 and v_4. Hence the number of distinct λ-colorings of C_4 in which v_2 and v_4 are colored the same is $\lambda\ (\lambda - 1)^2$.

If, on the other hand, v_2 and v_4 are colored differently, then there are $\lambda - 1$ choices for v_2 and $\lambda - 2$ choices for v_4. Since v_3 can be assigned any color except the two colors assigned to v_2

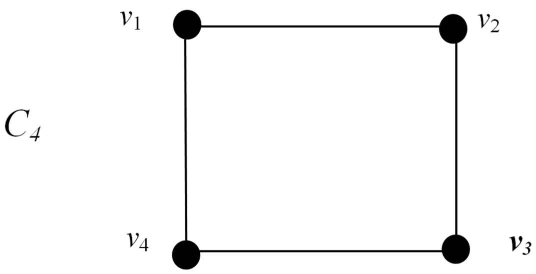

Figure 3. *The Chromatic polynomial of C_4.*

and v_4, the number of λ -colorings of C_4 in which v_2 and v_4 are colored differently is $\lambda\ (\lambda - 1)(\ \lambda - 2)^2$. Hence the number of distinct λ -colorings of C_4 is

$$\begin{aligned} P\ (C_4,\ \lambda\ ) &= \lambda\ (\lambda - 1)^2 + \lambda\ (\lambda - 1)(\ \lambda - 2)^2 \\ &= (\lambda - 1)\ (\lambda^2 - 3\ \lambda + 3) \\ &= \lambda^4 - 4\ \lambda^3 + 6\ \lambda^2 - 3\ \lambda \\ &= (\lambda - 1)^4 + (\lambda - 1). \end{aligned}$$

Theorem 2: The lead coefficient of $P\ (G, \lambda\ )$ is always 1.

Proof: The only partition that contributes to the lead coefficient is the one with n parts, giving an addend of $\lambda\ (\ \lambda - 1)...(\ \lambda - n + 1)$ where the coefficient of λ^{n} is 1.

Theorem 3: Let G be a graph of order n and size m. Then $P\ (G, \lambda\ )$ *is* a polynomial of degree n with leading coefficient 1 such that the coefficient of λ^{n-1} is $-$ m, and whose coefficients alternate in sign.

Proof: We proceed by induction on m. If $m = 0$, then $G = K_n$ and $P\ (G, \lambda) = \lambda^{n}$, as we have seen. Then $P\ (K_n, \lambda) = \lambda^{n}$ has the desired properties. Assume that the result holds for all graphs whose size is less than m, where $m \geq 1$. Let G be a graph of order m and let $e = uv$ an edge of G. By Corollary, Let G be a graph containing adjacent vertices u and v and let F be the graph obtained from G by identifying u and v. Then

$$P\ (G,\ ) = P\ (G - uv,\ ) - P\ (F, \lambda).$$

$$P\ (G, \lambda) = P\ (G - e, \lambda) - P\ (F, \lambda),$$

Where F is the graph obtained from G by identifying u and v. Since $G - e$ has order n and size $m - 1$, it follows by the induction hypothesis that

$$P\ (G - e, \lambda) = a_0\ \lambda^{n} + a_1\ \lambda^{n-1} + a_2\ \lambda^{n-2} + \cdots + a_{n-1}\ \lambda + a_n,$$

Where $a_0 = 1$, $a_1 = -(m - 1)$, $a_i \geq 0$ if i is even with $0 \leq i \leq n$, and $a_i \leq 0$ if i is odd with $1 \leq i \leq n$. Furthermore, since F has order $n - 1$ and size m', where $m' \leq m - 1$, it follows that

$$P\ (F, \lambda) = b_0\ \lambda^{n-1} + b_1\ \lambda^{n-2} + b_2\ \lambda^{n-3} + \cdots + b_{n-2}\ \lambda + b_{n-1},$$

where $b_0 = 1$, $b_1 = -m'$, $b_i \geq 0$ if i is even with $0 \leq i \leq n - 1$, and $b_i \leq 0$ if i is odd with $1 \leq i \leq n - 1$. Let G be a graph containing adjacent vertices u and v and let F be the graph obtained from G by identifying u and v. Then

$$P\ (G\ ) = P\ (G - uv,\ ) - P\ (F,\ ).$$

$$\begin{aligned} P\ (G, \lambda) &= P\ (G - e, \lambda) - P\ (F, \lambda) \\ &= (a_0\ \lambda^{n} + a_1\ \lambda^{n-1} + a_2\ \lambda^{n-2} + \cdots + a_{n-1}\ \lambda + a_n) - \\ &\quad (b_0\ \lambda^{n-1} + b_1\ \lambda^{n-2} + b_2\ \lambda^{n-3} + \cdots + b_{n-2}\ \lambda + b_{n-1}) \end{aligned}$$

$$= a_0 \lambda^n + (a_1 - b_0) \lambda^{n-1} + (a_2 - b_1) \lambda^{n-2} + \cdots$$
$$+ (a_{n-1} - b_{n-2})\_ + (a_n - b_{n-1}).$$

Since $a_0 = 1$, $a_1 - b_0 = -(m-1)-1 = -m$, $a_i - b_{i-1} \geq 0$ if i is even with $2 \leq i \leq n$, and $a_i - b_i - 1 \leq 0$ if i is odd with $1 \leq i \leq n$, $P(G, \lambda)$ has the desired properties and the theorem follows by mathematical induction.

Suppose that a graph G contains an end-vertex v whose only neighbor is u. Then, of course, $P(G - v, \lambda)$ is the number of λ -colorings of $G - v$. The vertex v can then be assigned any of the λ colors except the color assigned to u. This observation gives the following.

Theorem 4: If G is a graph containing an end-vertex v, then

$$P(G, \lambda) = (\lambda - 1) P(G - v, \lambda).$$

One consequence of this result is the following.

Corollary 5: if T is a tree of order $n \geq 1$, then

$$P(T, \lambda) = \lambda (\lambda - 1)^{n-1.}$$

Proof: We proceed by induction on n. For $n = 1$, $T = K_1$ and certainly $P(K_1, \lambda) = \lambda$. Thus the basis step of the induction is true. Suppose that $P(T', \lambda) = \lambda (\lambda - 1)^{n-2}$ for every tree T' of order $n - 1 \geq 1$ and let T be a tree of order n. Let v be an end-vertex of T. Thus $T - v$ is a tree of order $n - 1$. By induction hypothesis,

$P(T, \lambda) = (\lambda - 1) P(T - v, \lambda) = (\lambda - 1) [\lambda (\lambda - 1)^{n-2}] = \lambda (\lambda - 1)^{n-1}$, as desired [5].

2.6. Some Algebraic Properties of Chromatic Polynomials

Let G be a graph and λ be the set of colors to color G

1. The lead coefficient of P (G, λ) is always 1.
   a) The coefficient of λ^{n-1} in P (G, λ) is the negative of the number of edges.
2. The constant term, i.e. the coefficient of 1 in P (G, λ) is always zero.
3. The coefficient of λ in P (G, λ) is non-zero if and only if G is connected.
4. The coefficients of the chromatic polynomial alternate in sign. That is, for the coefficient a_m of λ^m we have $a_m \geq 0$ if $n \equiv m$ (2) and $a_m \leq 0$ otherwise.
5. The chromatic polynomial has no real root greater than n–1. Every two trees of the same order are chromatically equivalent. It is not known under what conditions two graphs are chromatically equivalent in general [5].

A graph G is chromatically unique if $P(H, \lambda) = P(G, \lambda)$ implies that $H \sim= G$. Here too it is not known under what conditions a graph is chromatically unique.

It has been conjectured by Read (1968) that the sequence of coefficients of any chromatic polynomial must first raise in absolute value and then fall, in other words, that no coefficient may be flanked by two coefficients having greater absolute value. However, even if true, this condition, together with the conditions of all the theorems above, would not be enough. Consider the polynomial $\lambda^5 - 11 \lambda^4 + 14 \lambda^3 - 6 \lambda^2 + 2 \lambda$. If this is the chromatic polynomial of a connected graph G, G should have five vertices and eleven edges. But the number of edges in a connected simple graph of order 5 is at least four and at most ten. So there is no graph for which this given polynomial is chromatic [6].

3. Applications of Vertex Colorings

Application 1

University course scheduling

Suppose that the vertices of a simple graph G represent the courses at a university in this model, two vertices are adjacent if and only if at least one student preregisters for both of the corresponding classes. Clearly, it would be undesirable for two such courses to be scheduled at the same time. Then the vertex-chromatic number $\chi(G)$ gives the minimum number of time periods in which to schedule the classes so that no student has a conflict between two courses [8].

Application 2

Allocation of channels to television and radio stations

Assume that there are k possible channels (frequencies) available for use by the n television stations in a certain country. As is well known, stations that are near to each other cannot use the same channel without causing interference. Thus, given any two stations, it may or may not be the case that they can use the same channel. The problem is to allocate a channel to each station in such a way that any two stations which need to have different channels get different channels.

Let us construct a graph G whose nodes represent the stations. We join two nodes by an edge if and only if the corresponding stations cannot use the same channel. Then any allocation of channels is, effectively, a coloring of G in k colors, and if it is proper then the condition about nearby stations being given different channel is satisfied. Thus the problem reduces to that of coloring a graph, and the chromatic polynomial will give the number of ways of allocating the k channels. If the vertices of a graph G represent radio stations, and two vertices are adjacent if the stations are close enough to interfere with each other, a coloring can be used to assign non-interfering frequencies to the stations [2].

Application 3

Separating combustible chemicals

Suppose that the vertices of a graph represent different kinds of chemicals needed in some manufacturing process. For each pair of chemicals that might explode if combined, there is an edge between the corresponding vertices. The chromatic number of this graph is the number of different storage areas required so that no two chemicals that mix explosively are mixed together [3].

Application 4

If the vertices of a graph represent traffic signals at an intersection, and two vertices are adjacent if the corresponding signals cannot be green at the same time, a coloring can be used to designate sets of signals that can be green at the same time [7].

4. Conclusion

A significant way of finding the minimum number of colors to color a graph is by means of vertex coloring. Scheduling has been a means of arranging project and time-tabling which allows participants to perform their activities without any clash. The structures are easily presented in the form of a graph. This also is applicable in communication networks (T. V. and Radio stations). Every simple graph has a chromatic polynomial; two different graphs can have the same polynomial. Not all polynomials are chromatic.

Among all essentials for human existence, the need to interact (on reliable and clear information) with others ranks just below our need to sustain life, likewise our daily activities are scheduled accordingly since some activities cannot be performed at the same time (Mutually exclusive). The need for proper time-tabling to avoid clash is as well very important. The advancement in communication technology that led to the invention of alternative communication channels uses the same mechanism to avoid interference for a reliable communication network. The use of vertex coloring also assists in ensuring this great achievement of stable networks. The chromatic polynomial of any Graph exists, we don't know yet under what condition two graphs will have the same chromatic polynomial in general.

5. Recommendation

Open problem: what are the general conditions for two graphs to have the same chromatic polynomial?

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